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The correspondence between multivariate spline ideals and piecewise algebraic varieties

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ABSTRACT

As a piecewise polynomial with a certain smoothness, the spline plays an important role in computational geometry. The algebraic variety is the most important subject in classical algebraic geometry. As the zero set of multivariate splines, the piecewise algebraic variety is a generalization of the algebraic variety. In this paper, the correspondence between piecewise algebraic varieties and spline ideals is discussed. Furthermore, Hilbert's Nullstellensatz for the piecewise algebraic variety is also studied.

1. Introduction

It is well known that, according to the classical Weierstrass theorem, any continuous function can be uniformly approximated by polynomials on a bounded domain. Therefore polynomials play an important role in approximation theory [1]. Unfortunately, the global property of polynomials is so strong that a polynomial can be determined solely by its properties on a neighborhood of a given point in the domain. However, splines, as piecewise polynomials, can be used to approximate any continuous, smooth, and even discontinuous function within any given tolerance [2]. Moreover, splines are easy to store, to evaluate, and to manipulate on digital computers. Splines have become fundamental tools of computational geometry, geometric modeling, numerical analysis, approximation theory, optimization, etc. [3–7,2]. Of central importance, perhaps, are univariate *B*-splines (or basic splines) first studied in some detail by Schoenberg in 1946 [8].

In 1975, Wang [5] pioneered the use of algebraic geometry in studying the theory of multivariate splines and discovered the fundamental theorem (Theorem 2.1) of multivariate splines, called the *smoothing cofactor-conformality* method. In a series of papers [9–12], Billera and Rose used the methods of homological and commutative algebra to study the algebraic properties and dimension of multivariate spline space, and the approach was further developed by Stiller, Schenck and Stillman [13–17]. Recently, Plaumann [4] studied the positivity of C^0 splines over a simplicial complex with the potential for application in optimization.

The *algebraic variety*, as the most important subject in classical algebraic geometry [18–21], is defined to be the intersections of hypersurfaces represented by multivariate polynomials. Because the objects are mainly represented by piecewise polynomials (splines), the *piecewise algebraic variety* defined as the intersection of surfaces represented by multivariate splines is a new topic in algebraic geometry and computational geometry. Moreover, studying the algebraic and geometric properties of the piecewise algebraic varieties is also important both in theory and in practice. For the recent researches on piecewise algebraic varieties, we refer the reader to [2,22–31].

The purpose of this paper is to introduce the basic algebro-geometric properties of spline ideals and piecewise algebraic varieties. In Section 2, we recall the smoothing cofactor-conformality method and some algebraic properties of multivariate splines. Next, piecewise algebraic varieties are presented in Section 3, followed by spline ideals and their properties in Section 4. Finally, the correspondence between piecewise algebraic varieties and spline ideals is studied in Section 5. Finally, we conclude the paper in Section 6.

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2. Multivariate spline space

Let $A^n := \{(a_1, \ldots, a_n) | a_i \in k, i = 1, \ldots, n\}$ be an n-dimensional affine space over an algebraically closed field k, and $D \subset A^n$ be a simply connected domain. A finite number of hyperplanes can be used to form a *partition* Δ on $D \subset A^n$. Then D is divided into a finite number of *cells*, $\delta_1, \ldots, \delta_T$. The boundary of each cell is called the *face* of Δ. Without loss of generality, we assume that D is a polyhedron in k^n. This means that Δ is a pure, hereditary polyhedral complex [32] and its cells are facets of Δ. The $(n-1)$-dimensional faces of Δ, $S_1, \ldots, S_E$, are called the *edges*. Let $l_i = 0$ be the equation of the affine hyperplane containing S_i. The zero-dimensional faces, $V_1, \ldots, V_Q$, are called *vertices* of Δ. If the face lies on the boundary of D, then it is called the *boundary face*; otherwise it is called an *interior face*.

Denote by $P_d(\Delta)$ the collection of piecewise polynomials of degree at most d

$$P_d(\Delta) := \{p | p_i = p|_{\delta_i} \in P_d, i = 1, 2, \ldots, T\},$$

where P_d is the space of n-variable polynomials of degree at most d. For an integer $0 \le \mu < d$, we say that

$$S_d^{\mu}(\Delta) := \{s | s \in C^{\mu}(D) \cap P_d(\Delta)\}$$

is a *multivariate spline space* with smoothness μ and total degree d over Δ. By using Bezout's theorem in algebraic geometry, Wang [5] discovered the following fundamental theorem on multivariate splines (for convenience, we consider bivariate spline space here).

Theorem 2.1 (*[5]*). *$s \in S_d^{\mu}(\Delta)$ if and only if the following conditions are satisfied:*

(1) *For each interior edge of Δ, which is defined by $S_i : l_i = 0$, there exists the so-called* smoothing cofactor *q_i such that*

$$s_{i1} - s_{i2} = l_i^{\mu+1} q_i,$$

where the polynomials s_{i1}, s_{i2} are determined by the restriction of s on the two cells δ_{i1} and δ_{i2} with S_i as the common edge and $q_i \in P_{\alpha-(\mu+1)}$, $\alpha = \max\{\deg(s_{i1}), \deg(s_{i2})\}$.

(2) *For any interior vertex V_j of Δ, the following* conformality conditions *are satisfied*

$$\sum [l_i^{(j)}]^{\mu+1} q_i^{(j)} \equiv 0,$$

where the sum runs over all the interior edges $S_i^{(j)} : l_i^{(j)} = 0$ passing through V_j, and the signs of the smoothing cofactors $q_i^{(j)}$ are fixed in such a way that when a point crosses $S_j^{(j)}$ from δ_{i2} to δ_{i1}, it goes around V_j in a counterclockwise manner.

More details about theory of multivariate splines using the smoothing cofactor-conformality method can be found in [5–7]. Billera and Rose [9–12,33] extended this method by using homological and commutative algebra to study the algebraic properties and dimension of $S_d^{\mu}(\Delta)$, and the approach was further developed by Stiller, Schenck and Stillman [13–17].

The space

$$S^{\mu}(\Delta) = \{s | s \in C^{\mu}(D) \cap P(\Delta)\}$$

is called the multivariate spline space with smoothness μ over Δ, where $P(\Delta)$ is the collection of piecewise polynomials over Δ. Obviously, $S_d^{\mu}(\Delta)$ is the subset of $s \in S^{\mu}(\Delta)$ such that the restriction of s to each cell in Δ is a polynomial of degree d or less. In fact, $S^{\mu}(\Delta)$ is a Nötherian ring [7,2]. Obviously, the polynomial ring $k[x_1, \ldots, x_n]$ is a subset of $S^{\mu}(\Delta)$, and it is a proper subset if partition Δ is generic.

Suppose that

$$B^{\mu}(\Delta) = \left\{ (g_1, \ldots, g_E) | \sum_{\delta \in c} g_\delta l_\delta^{\mu+1} = 0, \forall c \in C, g_i \in k[x_1, \ldots, x_n], i = 1, \ldots, E \right\},$$

where C denotes the set of cycles in the dual graph G_Δ of Δ. [32,33] presented the following algebraic meaning of Theorem 2.1 by considering a module $B^{\mu}(\Delta)$ built out of syzygies on the $l_i^{\mu+1}$.

Theorem 2.2 (*[32,33]*).

(1) *$S^{\mu}(\Delta)$ has the structure of a module over the ring $k[x_1, \ldots, x_n]$.*
(2) *$S_d^{\mu}(\Delta)$ is a finite-dimensional vector subspace of $S^{\mu}(\Delta)$.*
(3) *$S^{\mu}(\Delta)$ is isomorphic to $B^{\mu}(\Delta) \bigoplus k[x_1, \ldots, x_n]$ as a $k[x_1, \ldots, x_n]$-module.*
(4) *If G_Δ is a tree (i.e., a connected graph with no cycles), then $S^{\mu}(\Delta)$ is a free module for all $\mu \ge 0$.*

For each i, $1 \le i \le Q$, there is a unique function $X_i \in S_1^0(\Delta)$ such that

$$X_i(V_j) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

The X_i are called *Courant functions* of Δ. The *face ring* of Δ, denoted as $k[\Delta]$, is defined as the quotient ring $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal generated by square-free monomials not supported by faces of Δ. Billera [10] showed that in fact $S^0(\Delta)$ equals the algebra generated by the Courant functions over k.

Theorem 2.3 (*[10]*). *As k-algebras,*

$$S^0(\Delta) \cong k[\Delta]/\langle x_1 + \cdots + x_Q - 1 \rangle.$$

Following Sturmfels [34], in fact, I_Δ is the famous *Stanley–Reisner ideal* in combinatorics if Δ is a triangulation of D (which means that each cell of Δ is a simplex). Moreover, I_Δ is the radical of the initial ideal of toric ideals generated by a subset of vertices of Δ (see [34, Chap 8] for details). This is more interesting for studying the correspondence between splines and geometric combinatorics.

3. The piecewise algebraic variety

Definition 3.1 (*[7]*). Let Δ be a partition of region D. Suppose F is a subset of $S^\mu(\Delta)$. Then the set

$$X = \mathcal{Z}(F) := \{x \in D | f(x) = 0, \forall f \in F\} \tag{1}$$

is called a C^μ *piecewise algebraic variety* (PAV for short) over Δ.

If $f \in S^\mu(\Delta)$ and $X = \mathcal{Z}(f)$, then X is called a C^μ *piecewise hypersurface*. The following table shows the relationship of algebraic varieties and PAVs.

$$\begin{array}{ccc} \text{Polynomials} & \xrightarrow{\text{zero set}} & \text{Algebraic Variety} \\ \downarrow{\scriptstyle\text{piecewise}} & & \downarrow{\scriptstyle\text{piecewise}} \\ \text{Splines} & \xrightarrow{\text{zero set}} & \text{Piecewise Algebraic Variety} \end{array}$$

Studying PAVs is very important both in theory and in application. As a special case, the one-dimensional piecewise hypersurface is called a C^μ *piecewise algebraic curve* (PAC for short). The authors studied the Nöther-type theorem for C^0 PACs [25,29,30] and showed the application to the bivariate spline interpolation problem in [28]. In fact, most of the methods used in studying algebraic varieties cannot be used to study the PAVs since the spline ring $S^\mu(\Delta)$ is not the integral domain. Therefore, it is hard to study the PAVs. In recent years, Wang et al. have studied PAVs intensively [35,2,22–31]. We present some results here without proofs.

The *Zariski topology* on Δ is defined to be the topology whose closed sets are PAVs. For any PAV $V \subset D$, the Zariski topology on V is defined to be the subspace topology induced on V as a subset of D. From [7,24], it is a Nötherian topology, that is, any irreducible descent closed set sequence $X_1 \supset X_2 \supset \cdots$ is finite.

Definition 3.2. If a nonempty C^μ PAV V is expressed as the union of two nonempty proper PAVs V_1 and V_2, then V is called *reducible*; otherwise it is called *irreducible*.

Theorem 3.1 (*[7,24]*).
(1) *Each C^μ PAV can be expressed as the union of a finite number of irreducible C^μ PAVs.*
(2) *If $V, W \subset D$ are two C^μ PAVs over Δ, then so are $V \cap W$ and $V \cup W$.*

Definition 3.3. The *dimension* of a PAV V is defined as

$$\dim(V) := \max_{1 \le i \le T} \dim(\overline{V \cap \delta_i}), \tag{2}$$

where $\overline{V \cap \delta_i}$ is the Zariski closure of $V \cap \delta_i$ in the sense of the algebraic variety and $\dim(\overline{V \cap \delta_i})$ is the dimension of the algebraic variety $\overline{V \cap \delta_i}$.

For example, if V consists of a finite number of points, then $\dim(V) = 0$. For the topological space, the *Krull dimension* of V is the supremum of all integers m such that there exists a chain $V_0 \subset V_1 \subset \cdots \subset V_m$ of distinct irreducible closed subsets of V, denoted by $kr(X)$.

Theorem 3.2 (*[24]*). *Suppose V is a C^μ PAV. Then* $\dim(V) = kr(V)$.

In algebraic geometry, the hypersurface $\mathcal{Z}(f)$ is an $(n-1)$-dimensional algebraic variety for polynomial $f \in k[x_1, \ldots, x_n]$. But this may not hold for a piecewise hypersurface $f \in S^\mu(\Delta)$ because of the probability of $\mathcal{Z}(f|_{\delta_i}) \cap \delta_i = \emptyset$ for all $i = 1, \ldots, T$. So it does not make sense to define the piecewise hypersurface as the codimension 1 PAV. This explains in some sense why the PAV is a generalization of the algebraic variety.

4. Spline ideals

Definition 4.1. A subset I of $S^\mu(\Delta)$ is a *spline ideal* if it satisfies:

(1) $0 \in I$.
(2) If $f, g \in I$, then $f + g \in I$.
(3) If $f \in I$ and $h \in S^\mu(\Delta)$, then $hf \in I$.

Corollary 4.1 ([24]). *Suppose that $f_1, f_2, \ldots, f_r \in S^\mu(\Delta)$. Then the set*

$$\langle f_1, f_2, \ldots, f_r\rangle := \left\{\sum_{i=1}^{r} h_i f_i | \forall h_1, \ldots, h_r \in S^\mu(\Delta)\right\}$$

is a spline ideal of $S^\mu(\Delta)$, and it is called generated by $f_1, f_2, \ldots, f_r$.

Unfortunately, the spline ring $S^\mu(\Delta)$ is not an integral domain for arbitrary μ and Δ, i.e., there exist nonzero splines $f, g \in S^\mu(\Delta)$ such that $fg = 0$. This is the main reason why most of the methods used in studying polynomial ideals and algebraic varieties cannot be used to study the piecewise cases. Using the algebraic method and noticing that $S^\mu(\Delta)$ is a Nötherian ring, we see that each spline ideal $I \subset S^\mu(\Delta)$ is finitely generated [24], which is called the *Hilbert basis theorem* for multivariate spline space.

Theorem 4.1 ([24]). *Every spline ideal $I \subset S^\mu(\Delta)$ has a finite generating set. That is, $I = \langle f_1, f_2, \ldots, f_r\rangle$ for some $f_1, f_2, \ldots, f_r \in S^\mu(\Delta)$.*

If I is a spline ideal of $S^\mu(\Delta)$ generated by the finite spline set F, then $\mathcal{Z}(F) = \mathcal{Z}(I)$. It follows that $\mathcal{Z}(I)$ can be expressed as the common zeros of the finite generating set of F and the following result can be deduced using Theorem 4.1.

Corollary 4.2 ([24]). *The set $V \subset D$ is a C^μ PAV over Δ if and only if there exist finite number of splines $f_1, f_2, \ldots, f_r \in S^\mu(\Delta)$ such that*

$$V = \mathcal{Z}(f_1, f_2, \ldots, f_r) = \{x \in D | f_i(x) = 0, i = 1, 2, \ldots, r\}. \tag{3}$$

By Theorem 4.1, each ideal of $S^\mu(\Delta)$ is finitely generated, $I = \langle f_1, f_2, \ldots, f_r\rangle$. Hence, each PAV can be defined as $\mathcal{Z}(I) = \mathcal{Z}(f_1) \cap \cdots \cap \mathcal{Z}(f_r)$, which means that each PAV is the intersection of a finite number of piecewise hypersurfaces.

Definition 4.2 ([7,24]). A spline ideal $I \subset S^\mu(\Delta)$ is *prime* if whenever $f, g \in S^\mu(\Delta)$ and $fg \in I$, then either $f \in I$ or $g \in I$. Moreover, I is called *maximal* if $I \neq S^\mu(\Delta)$ and any ideal $J \subset S^\mu(\Delta)$ containing I is such that either $I = J$ or $J = S^\mu(\Delta)$.

Select any cell of Δ as the source cell. Then a *flow* curve $\overrightarrow{C}$ of Δ can be obtained as shown in [7], which starts from the source cell and goes through all other cells of Δ. The interior edge S_j passed through by $\overrightarrow{C}$ is called an *essential interior edge* with respect to $\overrightarrow{C}$ and denoted by $fr(S_j)$, the frontier of S_j in the flow $\overrightarrow{C}$. Let x be the n-indeterminate $(x_1, \ldots, x_n)$ and $S_j : l_j(x) = 0$ be an essential interior edge with respect to the flow $\overrightarrow{C}$. The *multivariate generalized truncated polynomial* is defined as

$$l_j(x)_+^{\mu+1} = \begin{cases} l_j(x)^{\mu+1}, & x \in fr(S_j). \\ 0, & x \in \Delta \setminus fr(S_j). \end{cases}$$

The structure of the maximal spline ideal was obtained in [24,31].

Theorem 4.2 ([24,31]). *Let $\overrightarrow{C}$ be a flow of Δ, δ_i ($i \in \{1, \ldots, T\}$) be the source cell, and $l_j(x) = 0$ be equations for the essential interior edges $S_j, j = 1, \ldots, r$, with respect to $\overrightarrow{C}$. Then the spline ideal I of $S^\mu(\Delta)$ is maximal if and only if there exists $t = (t_1, \ldots, t_n) \in \delta_i$ such that*

$$I = \langle x_1 - t_1, \ldots, x_n - t_n, l_1(x)_+^{\mu+1}, \ldots, l_r(x)_+^{\mu+1}\rangle.$$

Suppose that $f \in S^\mu(\Delta)$ and the exponent vector $\mathbf{m} = (m_1, \ldots, m_T) \in \mathbb{Z}_>^T$, where m_i are positive integers. Then $f^{\mathbf{m}} \in P(\Delta)$, where $f^{\mathbf{m}}|_{\delta_i} = (f|_{\delta_i})^{m_i}, i = 1, \ldots, T$. Obviously, $f^{\mathbf{m}}$ may not be a spline of $S^\mu(\Delta)$ for each $\mathbf{m} \in \mathbb{Z}_>^T$, since it may not satisfy the conditions of smoothing cofactor-conformality in Theorem 2.1.

Definition 4.3. A spline ideal $I \subset S^\mu(\Delta)$ is *radical* if there exists an exponent vector $\mathbf{m} \in \mathbb{Z}_>^T$ such that $f^{\mathbf{m}} \in I$ implies $f \in I$.

Definition 4.4. Let $I \subset S^\mu(\Delta)$ be a spline ideal. The *radical of I* is defined as the set

$$\{f \in S^\mu(\Delta) | f^{\mathbf{m}} \in I, \mathbf{m} \in \mathbb{Z}_>^T\},$$

denoted by $\sqrt{I}$.

The following lemma was presented in [28], which is useful for studying the radical of the spline ideal.

Lemma 4.1 ([28]). *If splines $f, g \in S^\mu(\Delta)$ and the piecewise polynomial $h \in P(\Delta)$ satisfy $f = gh$, then $h \in S^\mu(\Delta)$.*

Lemma 4.2. *If $I \subset S^\mu(\Delta)$ is a spline ideal, then $I \subset \sqrt{I}$ and $\sqrt{I} = \sqrt{\sqrt{I}}$.*

Proof. Since $f \in I$ implies $f^{(1,\dots,1)} \in I$, $f \in \sqrt{I}$ by Definition 4.4, which means that $I \subset \sqrt{I}$ and $\sqrt{I} \subset \sqrt{\sqrt{I}}$. Suppose that $f \in \sqrt{\sqrt{I}}$. Then there exists an exponent vector $\mathbf{m} \in \mathbb{Z}^T_>$ such that $f^{\mathbf{m}} \in \sqrt{I}$ by Definition 4.4. In the same way, $(f^{\mathbf{m}})^{\mathbf{n}} \in I$ for some exponent vector $\mathbf{n} \in \mathbb{Z}^T_>$. Suppose that $\mathbf{q} = \mathbf{m} \cdot \mathbf{n} = (m_1n_1, \dots, m_Tn_T)$, which means that $f^{\mathbf{q}} \in I$ and $\mathbf{q} \in \mathbb{Z}^T_>$. Therefore $f \in \sqrt{I}$. This yields $\sqrt{\sqrt{I}} \subset \sqrt{I}$ and proves that $\sqrt{I} = \sqrt{\sqrt{I}}$. $\square$

The following theorem shows that the radical of the spline ideal is radical too. We sketch the proof here and the details can be found in [31].

Theorem 4.3. *If $I \subset S^\mu(\Delta)$ is a spline ideal, then $\sqrt{I}$ is a radical spline ideal.*

Proof. Suppose that $f, g \in \sqrt{I}$. Then there exist two exponent vectors $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^T_>$ such that $f^{\mathbf{m}}, g^{\mathbf{n}} \in I$. Suppose that $\widehat{s} = \max\{m_1, \dots, m_T, n_1, \dots, n_T\}$, $s = 2\widehat{s}$, and the exponent vector $\mathbf{s} = (s, \dots, s) \in \mathbb{Z}^T_>$, where $\mathbf{m} = (m_1, \dots, m_T)$, $\mathbf{n} = (n_1, \dots, n_T)$. For cell δ_i, $1 \le i \le T$, we have

$$(f+g)^{\mathbf{s}}|_{\delta_i} = (f|_{\delta_i} + g|_{\delta_i})^s = \sum_{j=0}^{s} C_s^j (f|_{\delta_i})^j (g|_{\delta_i})^{s-j},$$

where the C_s^j are binomial coefficients. It follows that

$$\begin{aligned}(f+g)^{\mathbf{s}} &= \sum_{j=0}^{s} C_s^j f^{\mathbf{j}} g^{\mathbf{s}-\mathbf{j}} \\ &= \sum_{j=0}^{\widehat{s}-1} C_s^j f^{\mathbf{j}} g^{\mathbf{s}-\mathbf{j}} + \sum_{j=\widehat{s}}^{s} C_s^j f^{\mathbf{j}} g^{\mathbf{s}-\mathbf{j}} \\ &= \sum_{j=0}^{\widehat{s}-1} C_s^j f^{\mathbf{j}} \{g^{\mathbf{n}} g^{\mathbf{s}-\mathbf{j}-\mathbf{n}}\} + \sum_{j=\widehat{s}}^{s} C_s^j \{f^{\mathbf{m}} f^{\mathbf{j}-\mathbf{m}}\} g^{\mathbf{s}-\mathbf{j}},\end{aligned} \tag{4}$$

where $\mathbf{j} = (j, \dots, j) \in \mathbb{Z}^T_>$ and

$$\begin{aligned} f^{\mathbf{j}} &= f^{\mathbf{m}} f^{\mathbf{j}-\mathbf{m}}, j = \widehat{s}, \dots, s, \\ g^{\mathbf{s}-\mathbf{j}} &= g^{\mathbf{n}} g^{\mathbf{s}-\mathbf{j}-\mathbf{n}}, j = 0, \dots, \widehat{s}-1. \end{aligned}$$

Clearly, $f^{\mathbf{m}}, g^{\mathbf{n}} \in I \subset S^\mu(\Delta)$, $f^{\mathbf{j}} \in S^\mu(\Delta)$ for $j = \widehat{s}, \dots, s$, and $g^{\mathbf{s}-\mathbf{j}} \in S^\mu(\Delta)$ for $j = 0, \dots, \widehat{s}$. We get $f^{\mathbf{j}-\mathbf{m}}, g^{\mathbf{s}-\mathbf{j}-\mathbf{n}} \in S^\mu(\Delta)$ by Lemma 4.1. Therefore

$$\sum_{j=0}^{\widehat{s}-1} C_s^j f^{\mathbf{j}} \{g^{\mathbf{n}} g^{\mathbf{s}-\mathbf{j}-\mathbf{n}}\} \in I, \qquad \sum_{j=\widehat{s}}^{s} C_s^j \{f^{\mathbf{m}} f^{\mathbf{j}-\mathbf{m}}\} g^{\mathbf{s}-\mathbf{j}} \in I, \tag{5}$$

and $(f+g)^{\mathbf{s}} \in I$, which means that $f+g \in \sqrt{I}$. Furthermore, suppose that $f \in I$ and $h \in S^\mu(\Delta)$. Then $f^{\mathbf{m}} \in I$ for some exponent vector $\mathbf{q} \in \mathbb{Z}^T_>$. Suppose that $s = \max\{q_1, \dots, q_T\}$ and $\mathbf{s} = (s, \dots, s) \in \mathbb{Z}^T_>$. Then $f^{\mathbf{s}-\mathbf{m}} \in S^\mu(\Delta)$ since $f^{\mathbf{s}} = f^{\mathbf{s}-\mathbf{m}} f^{\mathbf{m}} \in S^\mu(\Delta)$. It follows that

$$(h \cdot f)^{\mathbf{s}} = h^{\mathbf{s}} f^{\mathbf{s}} = h^{\mathbf{s}} f^{\mathbf{s}-\mathbf{m}} f^{\mathbf{m}} \in I$$

since I is a spline ideal and $f^{\mathbf{m}} \in I$, $h^{\mathbf{s}} \in S^\mu(\Delta)$. Hence $h \cdot f \in \sqrt{I}$. This proves that $\sqrt{I}$ is a spline ideal in $S^\mu(\Delta)$ and the proof is completed by Lemma 4.2. $\square$

5. The correspondence between spline ideals and PAVs

The correspondence of spline ideals and PAVs is studied in this section, including Hilbert's Nullstellensatz for splines, which is a fundamental result identifying the spline ideals corresponding to PAVs.

Let V be a C^μ PAV over Δ. Then the set

$$\mathcal{I}(V) := \{f \in S^\mu(\Delta) | f(x) = 0, \forall x \in V\} \tag{6}$$

yields a spline ideal of $S^\mu(\Delta)$ and is called the *spline ideal of* V [24]. The following result shows the correspondence of spline ideals and PAVs in some sense [7,24].

Theorem 5.1 ([7,24]).

(1) *A maximal spline ideal of $S^\mu(\Delta)$ is prime.*
(2) *Let $V \subset D$ be a C^μ PAV. Then V is irreducible if and only if $\mathcal{I}(V)$ is a prime spline ideal of $S^\mu(\Delta)$.*
(3) *A maximal spline ideal of $S^\mu(\Delta)$ corresponds to a minimal irreducible closed subset of the C^μ PAV.*

By Corollary 4.2, V is a PAV if there exists some spline ideal $I \subset S^\mu(\Delta)$ such that $V = \mathcal{Z}(I)$. On the other hand, $\mathcal{I}(V)$ is a spline ideal of $S^\mu(\Delta)$ if V is a PAV. These give the correspondence between spline ideals and PAVs as

$$\begin{array}{ccccc} \text{Spline ideal} & \longrightarrow & \text{PAV} & \longrightarrow & \text{Spline ideal} \\ I & \longrightarrow & \mathcal{Z}(I) & \longrightarrow & \mathcal{I}(\mathcal{Z}(I)) \end{array}$$

Corollary 5.1. (1) *The map $\mathcal{Z}$: Spline ideals $\to$ PAVs is surjective.*
(2) *The map $\mathcal{I}$: PAVs $\to$ Spline ideals is injective.*

Proof. (1) By Corollary 4.2, each PAV is determined by a spline ideal, which means that the map $\mathcal{Z}$ is surjective. It is clear that two different spline ideals, like polynomial ideals, can determine the same PAV. This shows that the map $\mathcal{Z}$ is not a bijection.
(2) Obviously, for each PAV V, we have $\mathcal{Z}(\mathcal{I}(V)) = V$. If V_1, V_2 are two PAVs such that $\mathcal{I}(V_1) = \mathcal{I}(V_2)$, then

$$V_1 = \mathcal{Z}(\mathcal{I}(V_1)) = \mathcal{Z}(\mathcal{I}(V_2)) = V_2,$$

which means that the map $\mathcal{I}$ is injective. □

By the local property of splines, we see that $\mathcal{I}(\emptyset) = S^\mu(\Delta)$ does not hold and $\mathcal{Z}(I)$ may be empty although $I \subsetneq S^\mu(\Delta)$, which is the main difference between polynomial ideals and spline ideals. Therefore, Hilbert's Nullstellensatz for classical algebraic geometry does not hold for $S^\mu(\Delta)$. A natural question for spline ideals and PAVs arises: *that of, for the splines $f_1, f_2, \ldots, f_r \in S^\mu(\Delta)$, whether*

$$\langle f_1, f_2, \ldots, f_r\rangle = \mathcal{I}(\mathcal{Z}(f_1, \ldots, f_r)). \tag{7}$$

Unfortunately, the answer is not always positive. The following two corollaries give a preliminary answer for this question [31].

Corollary 5.2 (*[31]*)**.** *Let V be a PAV. For $f \in S^\mu(\Delta)$, if $f^{\mathbf{m}} \in \mathcal{I}(V)$, then $f \in \mathcal{I}(V)$, where $\mathbf{m} \in \mathbb{Z}^T_>$. Moreover, $\mathcal{I}(V)$ is a radical spline ideal.*

Corollary 5.3 (*[31]*)**.** *If the splines $f_1, f_2, \ldots, f_r \in S^\mu(\Delta)$ and a spline ideal $I \subset S^\mu(\Delta)$, then:*

(1) $\langle f_1, f_2, \ldots, f_r\rangle \subset \mathcal{I}(\mathcal{Z}(f_1, \ldots, f_r))$;
(2) $\sqrt{I} \subset \mathcal{I}(\mathcal{Z}(I))$.

Although using the above corollaries and the definition of the PAV may not lead to the conclusion that $\mathcal{I}(\mathcal{Z}(f_1, \ldots, f_r))$ equals $\langle f_1, \ldots, f_r\rangle$, the ideal of a PAV contains enough information for determining the PAV uniquely, which can be proved using elementary algebraic geometry.

Proposition 5.1 (*[31]*)**.** *If $V, W \subset D$ are two C^μ PAVs over Δ, then:*

(1) *$V \subset W$ if and only if $\mathcal{I}(V) \supset \mathcal{I}(W)$;*
(2) *$V = W$ if and only if $\mathcal{I}(V) = \mathcal{I}(W)$.*

It is known that the properties of splines (such as dimension and Bezout number) not only depend on the topological properties of the partition, but also sometimes depend on the geometric properties of the partition [7,2]. Then the problem (7) is not easy to solve. For some special cases, we can answer this question in some sense.

Theorem 5.2. *Suppose that $f \in S^\mu(\Delta)$ and the principle ideal $I = \langle f\rangle$. If $\mathcal{Z}(I) \cap \delta_i \neq \emptyset$ for $i = 1, \ldots, T$, then $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$.*

Proof. Obviously, $\sqrt{I} \subset \mathcal{I}(\mathcal{Z}(I))$ by Corollary 5.3. Suppose that $g \in \mathcal{I}(\mathcal{Z}(I))$; then $g|_{\delta_i} \in \mathcal{I}(\mathcal{Z}(f|_{\delta_i}))$ for $i = 1, \ldots, T$. By Hilbert's Nullstellensatz for algebraic varieties [18–20], for every $i \in \{1, \ldots, T\}$, there exists a positive integer r_i such that $g\,|^{r_i}_{\delta_i} = f|_{\delta_i} \cdot h|_{\delta_i}$, where $h|_{\delta_i}$ is a polynomial. Then we have $g^{\mathbf{r}} = fh$, where $\mathbf{r} = (r_1, \ldots, r_T) \in \mathbb{Z}^T_>$, $h \in P(\Delta)$. There must exist $\widetilde{\mathbf{r}} = \mathbf{r} + \widehat{\mathbf{r}}, \widehat{\mathbf{r}} \in \mathbb{Z}^T_>$ such that $g^{\widetilde{\mathbf{r}}} \in S^\mu(\Delta)$ and $g^{\widetilde{\mathbf{r}}} = f\widetilde{h}$, where $\widetilde{h} \in P(\Delta)$ (the vector $\widetilde{\mathbf{r}} = (r, \ldots, r)$ satisfies this condition, where $r = \max\{r_1, \ldots, r_T\}$). By Lemma 4.1, we have $\widetilde{h} \in S^\mu(\Delta)$, which means that $g \in \sqrt{I}$ and $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$. □

Definition 5.1. Suppose that $f, g \in P(\Delta)$. If two polynomials $f_i = f|_{\delta_i}$ and $g_i = g|_{\delta_i}$ have no non-constant common factors for all $i \in \{i = 1, \ldots, T\}$, we say that they *have no local common factors*.

We consider the two-dimensional problem and suppose that $n = 2$ in the rest of the paper. Let Δ_O be the partition with only one interior vertex O and Δ_c be the *cross-cut partition* consisting of finite straight lines, which are called cross-cut lines. The edge segments of the partition starting from the interior vertices and ending on the boundary vertices are called rays. A partition is said to be a *quasi-cross-cut partition* if each interior edge is either a part of a cross-cut line or a part of a ray, denoted by Δ_{qc}. Let partition Δ be one of Δ_O, Δ_c, and Δ_{qc} in the rest of the paper. The following *Nöther-type theorem* shows a fundamental result for PACs.

Theorem 5.3 (*[25,28–30]*)**.** *Let l, m, r be natural numbers. Suppose that the splines $f \in S_l^\mu(\Delta), g \in S_m^\mu(\Delta), h \in S_r^\mu(\Delta)$ are such that f, g have no local common factors. If two PACs $\mathcal{Z}(f), \mathcal{Z}(g)$ meet exactly at mlT distinct points and the PAC $\mathcal{Z}(h)$ passes through these mlT distinct points, then*

$$h = uf + vg,$$

where $u \in S_{r-l}^\mu(\Delta), v \in S_{r-m}^\mu(\Delta)$.

Let $\deg(f) = \max\{\deg(f|_{\delta_i}) | i = 1, \ldots, T\}$ be the degree of $f \in S^\mu(\Delta)$. By Theorem 5.3, we get the following Hilbert Nullstellensatz for PACs directly.

Theorem 5.4 (*Hilbert's Nullstellensatz*)**.** *Suppose that the splines $f_1, f_2 \in S^\mu(\Delta)$ have no local common factors, and are such that two PACs $\mathcal{Z}(f_1), \mathcal{Z}(f_2)$ meet exactly at $\deg(f_1)\deg(f_2)T$ distinct points and ideal $I = \langle f_1, f_2\rangle$. Then $f \in \mathcal{I}(\mathcal{Z}(I))$ if and only if $f \in I$. Moreover, in this sense, $I = \mathcal{I}(\mathcal{Z}(I))$.*

Theorem 5.5 (*Hilbert's Nullstellensatz*)**.** *Suppose that the splines $f_1, f_2 \in S^\mu(\Delta)$, $\mathcal{Z}(f_1|_{\delta_i}, f_2|_{\delta_i}) \neq \emptyset$, $\mathcal{Z}(f_1|_{\delta_i}, f_2|_{\delta_i}) \subset \delta_i$ for all $1 \le i \le T$, and the ideal $I = \langle f_1, f_2\rangle$. Then $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$.*

Proof. Obviously, $\sqrt{I} \subset \mathcal{I}(\mathcal{Z}(I))$ by Corollary 5.3. Suppose $g \in \mathcal{I}(\mathcal{Z}(I))$; then $g|_{\delta_i} \in \mathcal{I}(\mathcal{Z}(f_1|_{\delta_i}, f_2|_{\delta_i}))$ for $i = 1, \ldots, T$. By Hilbert's Nullstellensatz [18–20], for every $i \in \{1, \ldots, T\}$ there exists some positive integer r_i such that $g|_{\delta_i}^{r_i} = f_1|_{\delta_i} \cdot h_1|_{\delta_i} + f_2|_{\delta_i} \cdot h_2|_{\delta_i}$, where $h_1|_{\delta_i}, h_2|_{\delta_i}$ are polynomials on cell δ_i. Then we have $g^{\mathbf{r}} = f_1 h_1 + f_2 h_2$, where $\mathbf{r} = (r_1, \ldots, r_T) \in \mathbb{Z}_>^T$, $h \in P(\Delta)$. There must exist $\widetilde{\mathbf{r}} = \mathbf{r} + \widehat{\mathbf{r}}, \widehat{\mathbf{r}} \in \mathbb{Z}_>^T$, such that $g^{\widetilde{\mathbf{r}}} \in S^\mu(\Delta)$ and $g^{\widetilde{\mathbf{r}}} = f_1\widetilde{h_1} + f_2\widetilde{h_2}$, where $\widetilde{h_1}, \widetilde{h_2} \in P(\Delta)$ (the vector $\widetilde{\mathbf{r}} = (r, \ldots, r)$ satisfies this condition, where $r = \max\{r_1, \ldots, r_T\}$). By Theorem 5.6, we have $g^{\widetilde{\mathbf{r}}} \in I$, which means that $g \in \sqrt{I}$ and $\sqrt{I} = \mathcal{I}(\mathcal{Z}(I))$. □

Theorem 5.6. *Suppose that $f, g \in S^\mu(\Delta)$ and $I = \langle f, g\rangle$. If $\mathcal{Z}(f, g)$ has no points lying on the interior edges and $h^{\mathbf{m}} = af + bg$, where $\mathbf{m} \in \mathbb{Z}_>^T$, $a, b \in P(\Delta)$, then $h^{\mathbf{m}} \in I$. In this sense, there exist splines $u, v \in S^\mu(\Delta)$ such that $h^{\mathbf{m}} = uf + vg$.*

Proof. The assumption of the theorem gives the relation $h^{\mathbf{m}} = af + bg$. From the proof of Theorem 5.3 presented in [25, 28–30], we get $h^{\mathbf{m}} = uf + vg$ in the same way, where $u, v \in S^\mu(\Delta)$. We omit the details of the proof here. □

6. Conclusions

As the zero set of multivariate splines, the PAV is a new topic in algebraic geometry and computational geometry. Studying the algebraic and geometric properties of the PAVs is also important both in theory and in practice. In this paper, the correspondence between PAVs and spline ideals is studied. Furthermore, Hilbert's Nullstellensatz for the PAV is also discussed. The Nöther-type theorem for an arbitrary partition and how to compute the dimension for a PAV remain open problems. We will study these two problems and algebraic and geometric properties of PAVs in future.

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References

[1] R.H. Wang, C.J. Li, C.G. Zhu, A Course for Computational Geometry, Science Press, Beijing, 2008.
[2] R.H. Wang, Multivariate spline and algebraic geometry, J. Comput. Appl. Math. 121 (2000) 153–163.
[3] C. de Boor, A Practical Guide to Splines, Springer-Verlag, Berlin/New York, 1978.
[4] D. Plaumann, Positivity of continuous piecewise polynomials, http://arxiv.org/abs/1004.3104.
[5] R.H. Wang, The structural characterization and interpolation for multivariate splines, Acta Math. Sinica, Engl. Ser. Mar. 18 (1975) 10–39.
[6] R.H. Wang, The dimension and basis of spaces of multivariate spline, J. Comput. Appl. Math. 12 (13) (1985) 163–177.
[7] R.H. Wang, Multivariate Spline Functions and Their Applications, Science Press/Kluwer Pub., New York/Beijing, 2001.
[8] I.J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math. 4 (45–99) (1946) 112–141.
[9] L. Billera, Homology of smooth splines: generic triangulations and a conjecture of Strang, Trans. Amer. Math. Soc. 310 (1988) 325–340.
[10] L. Billera, The algebra of continuous piecewise polynomials, Adv. Math. 76 (1989) 170–183.
[11] L. Billera, L. Rose, A dimension series for multivariate splines, Discrete Comput. Geom. 6 (1991) 107–128.
[12] L. Billera, L. Rose, Modules of piecewise polynomials and their freeness, Math. Z. 209 (1992) 485–497.
[13] Y. Ko, Dimensions of Bivariate Spline Spaces and Algebraic Geometry, Ph.D. Thesis, Texas A&M University, 2009.
[14] W. Lau, P. Stiller, Splines and algebraic geometry, preprint.
[15] H. Schenck, A spectral sequence for splines, Adv. Appl. Math. 19 (1997) 183–199.
[16] H. Schenck, M. Stillman, A family of ideals of minimal regularity and the Hilbert series of $C^r(\widehat{\Delta})$, Adv. Appl. Math. 19 (1997) 169–182.
[17] H. Schenck, M. Stillman, Local cohomology of bivariate splines, J. Pure Appl. Algebra 117 (118) (1997) 535–548.

[18] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms, 2nd ed., Springer-Verlag, New York, 1996.
[19] R. Hartshorn, Algebraic Geometry, Springer-Verlag, New York, 1977.
[20] D. Perrin, Algebraic Geometry: An Introduction, Springer-Verlag, London, 2008.
[21] R.J. Walker, Algebraic Curves, Princeton: Princeton University Press, 1950.
[22] R.H. Wang, Recent researches on multivariate spline and piecewise algebraic variety, J. Comput. Appl. Math. 221 (2008) 460–471.
[23] R.H. Wang, On Piecewise algebraic variety, in: Third International Congress of Chinese Mathematicians, Part 1, 2, in: AMS/IP Stud. Adv. Math., vol. 42, Amer. Math. Soc., Providence, 2008, pp. 297–306.
[24] R.H. Wang, C.G. Zhu, Piecewise algebraic varieties, Prog. Nat. Sci. 14 (2004) 568–572.
[25] R.H. Wang, C.G. Zhu, Nöther-type theorem of piecewise algebraic curves, Prog. Nat. Sci. 14 (2004) 309–313.
[26] R.H. Wang, C.G. Zhu, Cayley–Bacharach theorem of piecewise algebraic curves, J. Comput. Appl. Math. 163 (2004) 269–276.
[27] C.G. Zhu, R.H. Wang, Piecewise semialgebraic sets, J. Comput. Math. 23 (2005) 503–512.
[28] C.G. Zhu, R.H. Wang, Lagrange interpolation by bivariate splines on cross-cut partitions, J. Comput. Appl. Math. 195 (2006) 326–340.
[29] C.G. Zhu, R.H. Wang, Nöther-type theorem of piecewise algebraic curves on triangulation, Sci. China Ser. A: Math. 50 (2007) 1227–1232.
[30] C.G. Zhu, R.H. Wang, Nöther-type theorem of piecewise algebraic curves on quasi-cross-cut partition, Sci. China Ser. A: Math. 52 (2009) 701–708.
[31] C.G. Zhu, R.H. Wang, Algebra–geometry of piecewise algebraic varieties, (submitted for publication).
[32] D. Cox, J. Little, D. O'Shea, Using Algebraic Geometry, Springer-Verlag, New York, 1998.
[33] L. Rose, Combinatorial and topological invariants of modules of piecewise polynomials, Adv. Math. 116 (1995) 34–45.
[34] B. Sturmfels, Gröbner Bases and Convex Polytopes, Providence: Amer. Math. Soc. (1995).
[35] X.Q. Shi, R.H. Wang, Bezout's number for piecewise algebraic curves, BIT 39 (1999) 339–349.